ST402 Principals and Methods of Statistical Practice
Mid-term Test

Answer as much of this as you can in the 45 minutes available. You may consult your notes but not your colleagues. All questions will be marked out of 10 (although question 4 may be slightly longer than the others).

1. I have an urn with 3 disks of which 2 are green and 1 is red. I draw 2 discs from the urn without replacement; X is the number of green discs that I draw. I replace the discs in the urn, and draw another X discs from the urn, without replacement. Suppose that the X discs drawn contain Y green discs.

   (a) Find $E(X)$ and $E(Y|X)$. Hence find $E(Y)$.

   (b) Find the probability mass function of Y.

2. Let X and Y be random variables with joint density function

$$f_{X,Y}(x,y) = \begin{cases} k(x+y), & 0 < x < 1, 0 < y < 1, 0 < x+y < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

   (a) Find k.

   (b) Find the density of $X|Y = y$ and hence find $E(X|Y = \frac{1}{2})$.

3. Let Y and Z be independent random variables with exponential distributions with the same mean $\frac{1}{\lambda}$.

   (a) Find the joint density function of (Y, Z).

   (b) Find the Jacobian of the transformation $U = Z + Y$, $V = Z/Y$.

   (c) find the joint density function of (U, V), and show that U,V are independent.

4. Let

$$X = \begin{cases} \Sigma_{i=1}^{N} Y_i & N > 0 \\ 0 & N = 0 \end{cases}$$

where N is a discrete random variable taking values from the set of non-negative integers, and $Y_1, Y_2, \ldots$ is a sequence of i.i.d. random variables that are independent of N such that $P(Y_1 = 1) = P(Y_1 = 0) = 0.5$.

(a) Let N be a Poisson random variable with mean λ. Find $E(X)$ and the moment generating function (mgf) of X. Thus find $P(X = k)$ for all k.

(b) Now let N have a geometric distribution such that $P(N = n) = (0.5)^{n+1}$ for $n = 0, 1, 2, \ldots$Find $E(X)$, the mgf of X and prove that

$$P(X = k) = \frac{2}{3}\left(\frac{1}{3}\right)^k, \quad k = 0, 1, 2, \ldots.$$

(You may use the fact that the mgf of a Poisson random variable is $e^{(\lambda(e^t-1))}$.)